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Stability of Kolmogorov scaling in anisotropically forced turbulence

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Renormalization-group analysis of randomly stirred fluid with anisotropic distribution of random force is carried out at one-loop order. The axial anisotropy is introduced by free parameters of external forcing in the Navier-Stokes equation, but the anisotropy parameters are not assumed to be small. The region of stability of the Kolmogorov scaling regime in the space of anisotropy parameters has been determined for several space dimensionalities $2<d\leqslant 3$. The Kolmogorov constant and the amplitudes of longitudinal and transverse projection operators with respect to the preferred direction in the energy spectrum have been calculated in situations where the competition between the anisotropy of the external forcing and the Navier-Stokes dynamics may affect the stability of the Kolmogorov regime. Extension to more complex magnetohydrodynamic systems is under investigation. [S1063-651X(97)00501-1]

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I. INTRODUCTION

Nonlinearity of the Navier-Stokes equation for high Reynolds numbers makes the theoretical description of developed turbulence very difficult. Since the 1970s there has been a significant growth of interest in the theoretical investigation of statistical hydrodynamic models, where the phenomenological statistics of an external random forcing have been used for the modeling of very complex flow instabilities. In the present paper the stochastic variant of the Navier-Stokes equation and the magnetohydrodynamic (MHD) equations are discussed in the case when the axial symmetry of external random forcing has been taken into account.

The deviation of the statistical behavior of the fully developed turbulence from the isotropic statistics was confirmed by a variety of experiments and computer simulations, and the role of spatially oriented fluctuations is a permanently discussed aspect of turbulence physics.

This deviation may be induced by the presence of specific initial or boundary conditions, interactions of fluctuating fields with mean flow gradients, or external fields. In the context of measurement the anisotropy contributions could also be relevant for the experimental errors in determination of the turbulent energy dissipation [1]. The comprehensive paper of Herring [2] devoted to theoretical study of anisotropic splitting of kinetic energy spectra using direct interaction approximation could be considered the starting point for the development of this particular direction of research.

It is a widespread opinion that the local isotropy postulate of Kolmogorov together with the assumption that developed turbulence in the inertial range is independent of the viscous cutoff, shape, and size of boundary conditions, creates only a rough basis for the satisfactory understanding of the full complexity of the turbulence. Therefore the advanced statistical hydrodynamics tries to give a more accurate answer to the question of the validity of the classical phenomenology, which ignores the effect of anisotropy inside the inertial subscales. This question has been treated earlier [3,4] with the assumption that the anisotropy in the stochastic forcing is small. In the present paper we have carried out a one-loop renormalization-group analysis of anisotropic stochastic turbulence without this assumption, and hope that our results shed more light on the problem of validity of the classical phenomenology.

Another important motivation of this work comes from the study of weakly anisotropic stochastic magnetohydrodynamics [5]. The results show that even small anisotropy of the forcing leads to a large-scale modification of viscous and resistive behavior, viz., to the asymptotical growth of large-scale Lorentzian terms. The dominant effective Lorentzian forces have been found to lead to the supression of the Kolmogorov scaling. In the present work a new approach is proposed for the analysis of the stability of the critical regimes of the stochastic axisymmetric MHD.

II. RENORMALIZATION OF THE STOCHASTIC NAVIER-STOKES EQUATION

Turbulent flow may be described by a random velocity field $\vec{v}(\vec{x},t)$ ($\vec{v}$ and $\vec{x}$ are d-dimensional vectors), whose evolution is governed by the randomly forced Navier-Stokes equation

$$\partial_t\vec{v}+\hat{P}(\vec{v}\cdot\vec{\nabla})\vec{v}-\nu\vec{\nabla}^2\vec{v}-\vec{f}^A=\vec{f},\quad \vec{\nabla}\cdot\vec{v}=\vec{\nabla}\cdot\vec{f}=0\,, \tag{2.1}$$

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where ν is the kinematic viscosity and $\hat{P}$ is the transverse projection operator, defined as $\hat{P}=\hat{I}-\nabla^{-2}\vec{\nabla}\otimes\vec{\nabla}$ ($\hat{I}$ is the $d\times d$ unit matrix). The explicit form of the anisotropic dissipative term $\vec{f}^A$ will be specified later. Following the tradition of stochastic models of turbulence, the randomness in Eq. (2.1) is introduced by the large-scale random forcing $\vec{f}(\vec{x},t)$ with Gaussian statistics defined by the averages

$$\langle f_j\rangle=0\,,\quad \langle f_j(\vec{x}_1,t_1)f_s(\vec{x}_2,t_2)\rangle=D_{js}(\vec{x}_1-\vec{x}_2,t_1-t_2).$$

Here, the two-point correlation matrix

$$D_{js}(\vec{x},t)=\delta(t)\int\frac{d^d\vec{k}}{(2\pi)^d}\tilde{\mathcal{D}}_{js}(\vec{k})\exp[i\,\vec{k}\cdot\vec{x}] \tag{2.2}$$

can be parametrized [3,5] as

$$\tilde{\mathcal{D}}_{js}(\vec{k})=g_v\nu^3\Lambda^{2\epsilon}k^{4-d-2\epsilon}\{[1+\alpha_1\xi_k^2]P_{js}(\vec{k})+\alpha_2R_{js}(\vec{k})\} \tag{2.3}$$

for d-dimensional anisotropic random force. The matrices P and R of transverse projection operators in the wave-number space are defined by the relations

$$P_{js}(\vec{k})=\delta_{js}-\frac{k_jk_s}{k^2},$$

$$R_{js}(\vec{k})=\left(n_j-\xi_k\frac{k_j}{k}\right)\left(n_s-\xi_k\frac{k_s}{k}\right),$$

$$\xi_k=\frac{\vec{k}\cdot\vec{n}}{k}. \tag{2.4}$$

In the expressions (2.3) and (2.4) $\vec{k}$ denotes the wave vector and the unit vector $\vec{n}$ yields the direction of the anisotropy axis. In the definition (2.3) the most general parametrization of the nonhelical D_{js} tensor, which depends on the dimensionless free parameters α_1 and α_2 maintaining the property of incompressibility, is used. The force correlation tensor D_{js} must be positive definite, which leads to the following restrictions to the values of the parameters $\alpha_1\geqslant-1$, $\alpha_2\geqslant0$. For nonzero α_1, α_2 the forcing describes differences in energy injection in the preferred direction and directions perpendicular to it with the subsequent generation of anisotropic structures in the large-scale eddies. Irrespective of the precise details of large-scale dynamics a successive isotropization of these structures towards the small scales can be expected. We emphasize that, in contrast with the previous work [3,4], these parameters are not considered small in the present analysis. The parameter Λ with the wave-number dimension is a scale setting parameter characterizing the size of the microeddy region, and g_v is a positive dimensionless constant. The usual assumption of stochastic theories exhibiting the Kolmogorov scaling behavior is that the forcing is localized near the origin in the wave-vector space, therefore the physical value of the parameter ϵ is $\epsilon=2$.

Measurable quantities in the analysis of the developed turbulence are the correlation functions of velocity, which may be calculated by means of the functional integral

$$\left\langle\prod_{j=1}^N v_{m_j}(\vec{x}_j,t_j)\right\rangle=\int[d^d\vec{v}][d^d\tilde{\vec{v}}]\mathcal{P}(\vec{v},\tilde{\vec{v}})\times\prod_{j=1}^N v_{m_j}(\vec{x}_j,t_j),\quad 1\leqslant m_j\leqslant d. \tag{2.5}$$

Following [7], we write the probability distribution of the velocity field $\vec{v}$ in the form $\mathcal{P}=\text{norm}\times\exp[S]$ with the action

$$\begin{aligned}S=&\int d^d\vec{x}_1dt_1d^d\vec{x}_2dt_2\\&\times\left[\frac{1}{2}\tilde{v}_j(\vec{x}_1,t_1)D_{js}(\vec{x}_1-\vec{x}_2,t_1-t_2)\tilde{v}_s(\vec{x}_2,t_2)\right]\\&+\int d^d\vec{x}dt\{\tilde{\vec{v}}(\vec{x},t)[-\partial_t\vec{v}-(\vec{v}\cdot\vec{\nabla})\vec{v}\\&+\nu\vec{\nabla}^2\vec{v}+\vec{f}^A]_{(\vec{x},t)}\}.\end{aligned} \tag{2.6}$$

Note that a new independent of the $\vec{v}$ auxiliary incompressible field $\tilde{\vec{v}}$ has been introduced in the transformation of the stochastic problem (2.1)–(2.4) into the functional form. By means of the renormalization-group method it is possible to extract large-scale asymptotics of the correlation functions. To this end, the field theory (2.6) must first be renormalized, which is most conveniently carried out by multiplicative renormalization [6].

In order to obtain a multiplicatively renormalizable theory [7] it is necessary to include in the action terms corresponding to all superficially divergent one-particle irreducible correlation functions of both fields $\tilde{\vec{v}}$ and $\vec{v}$. The analysis of the divergences of the correlation functions shows that to have a multiplicatively renormalizable theory, we must take the anisotropic dissipative term in the Navier-Stokes equation (2.1) in the form

$$\begin{aligned}\vec{f}^A=\nu[&\chi_1(\vec{n}\cdot\vec{\nabla})^2\vec{v}+\chi_2\hat{P}\vec{n}\vec{\nabla}^2(\vec{n}\cdot\vec{v})\\&+\chi_3\hat{P}\vec{n}(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v})].\end{aligned} \tag{2.7}$$

Here, χ_1,χ_2, and χ_3 are dimensionless parameters describing the relative impact of the different anisotropic structures on the viscous dissipation. Technically, these new parameters are required to bring about the cancellation of the divergences in the diagrammatic expression of the correlation functions (2.5). Physically, the expression (2.7) for the anisotropic force term in the Navier-Stokes equation (2.1) may be obtained on phenomenological grounds in the same fashion as the viscous terms in the isotropic Navier-Stokes equation, where they are presented as the divergence of the viscous stress tensor [8]. Constructing the most general expression of the phenomenological viscous stress tensor in the presence of a preferred direction and taking its divergence we arrive at the expression (2.7) for the anisotropic solenoidal part of the dissipative term of the Navier-Stokes equation. The coefficients χ_i are the dimensionless coefficients of viscosity of the anisotropic part of the viscous stress tensor, and represent the orientational redistribution of the momentum flux accompanied by energy dissipation.

In the leading order in the expansion parameter g_v the contributions to the renormalization of the stochastic Navier-Stokes equation (2.1) with the anisotropic force $\vec{f}^A$ in the form (2.7) are extracted, according to the standard rules of the minimal subtraction method [9], from the one-loop irreducible Feynman diagram, whose analytic expression is

$$\Gamma_{ab}^{(2)}(\vec{k})=\int\frac{d^d\vec{p}}{(2\pi)^d}\int_{-\infty}^{\infty}dt\Delta_{\alpha\beta}(\vec{p},t)\widetilde{\Delta}_{\gamma\delta}(\vec{k}-\vec{p},t)\times V_{a\alpha\gamma}(\vec{k})V_{\delta b\beta}(\vec{k}-\vec{p}). \tag{2.8}$$

These terms are singular in the limit $\epsilon\to 0$ with the tensor structures

$$O(k^2\delta_{ab}/\epsilon),\quad O((\vec{k}\cdot\vec{n})^2\delta_{ab}/\epsilon),$$

$$O(k^2n_an_b/\epsilon),\quad O((\vec{k}\cdot\vec{n})^2n_an_b/\epsilon).$$

The elements of the perturbation expansion appearing in the integral (2.8) are obtained from the quadratic part and the cubic term $\tilde{\vec{v}}\vec{v}\vec{v}$ of the action (2.6). In the $(\vec{k},t)$ representation the propagators $\widetilde{\Delta}$, Δ, and the symmetrized vertex V may be written in the form

$$\Delta_{\alpha\beta}(\vec{k},t)=\frac{g_v}{2}\nu^2\Lambda^{2\epsilon}k^{2-d-2\epsilon}\{w_1(\xi_k,\tau_k)P_{\alpha\beta}(\vec{k})+w_2(\xi_k,\tau_k)R_{\alpha\beta}(\vec{k})\},$$

$$\widetilde{\Delta}_{\alpha\beta}(\vec{k},t)=\theta(t)\{w_3(\xi_k,\tau_k)P_{\alpha\beta}(\vec{k})+w_4(\xi_k,\tau_k)R_{\alpha\beta}(\vec{k})\},$$

$$V_{\alpha\beta\gamma}(\vec{k})=i\{P_{\alpha\beta}(\vec{k})k_\gamma+P_{\alpha\gamma}(\vec{k})k_\beta\}, \tag{2.9}$$

where

$$w_1=\frac{1+\alpha_1\xi^2}{A}e^{-|\tau|A},\quad \tau=\nu k^2t,$$

$$w_2=\frac{1}{A}\left[\alpha_2e^{-|\tau|A}+[1+\alpha_1+(\alpha_2-\alpha_1)(1-\xi^2)]\times\frac{1}{B(1-\xi^2)}(Ae^{-|\tau|B}-Be^{-|\tau|A})\right],$$

$$w_3=e^{-\tau A},\quad w_4=-\frac{1}{1-\xi^2}(e^{-\tau B}-e^{-\tau A}),$$

$$A=1+\chi_1\xi^2,\quad M=\chi_2+\chi_3\xi^2,\quad B=A+(1-\xi^2)M. \tag{2.10}$$

In a multiplicatively renormalizable model the renormalization leads to multiplication of the parameters of the model by renormalization constants Z. The correlation functions of the renormalized model are calculated with the use of the renormalized action, which in the present model is of the form

$$S^R=\int d^d\vec{x}_1dt_1d^d\vec{x}_2dt_2\times\left[\frac{1}{2}\tilde{v}_j(\vec{x}_1,t_1)D_{js}(\vec{x}_1-\vec{x}_2,t_1-t_2)\tilde{v}_s(\vec{x}_2,t_2)\right]+\int d^d\vec{x}dt(\tilde{\vec{v}}(\vec{x},t)\{-\partial_t\vec{v}-(\vec{v}\cdot\vec{\nabla})\vec{v}+\nu[Z_1\vec{\nabla}^2\vec{v}+Z_2\chi_1(\vec{n}\cdot\vec{\nabla})^2\vec{v}+Z_3\chi_2\vec{n}\vec{\nabla}^2(\vec{n}\cdot\vec{v})+Z_4\chi_3\vec{n}(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v})]\}). \tag{2.11}$$

The renormalization constants Z_j ($Z_j\sim 1/\epsilon$ at the leading order of the expansion in g_v) may be constructed in such a way that they give rise to terms which cancel the divergences appearing in the integrals of the model (2.11). It should be noted that renormalization gives rise to the renormalized action (2.11) for space dimensions $d>2$ only. At two dimensions new divergences appear, which lead to the renormalization of the term quadratic in the auxiliary field $\tilde{\vec{v}}$. This results in additional complications in the renormalization-group analysis of the model near two dimensions [10], therefore we restrict the present treatment to space dimensions $d>2$.

In the leading order the renormalization constants are extracted from the divergent part of the integral (2.8) according to the following prescription:

$$\Gamma_{ab}^{(2)}(\vec{k})=-\nu k^2\delta_{ab}(1-Z_1)-\nu(\vec{k}\cdot\vec{n})^2\delta_{ab}(1-Z_2)-\nu k^2n_an_b(1-Z_3)-\nu(\vec{k}\cdot\vec{n})^2n_an_b(1-Z_4)+\text{regular terms}. \tag{2.12}$$

Rather complicated explicit expressions for the renormalization constants can be found in Appendix A.

III. STABILITY OF ANISOTROPIC HYDRODYNAMIC THEORY

In the renormalization-group method the correlation functions (2.5) are expressed in terms of scaling functions containing effective variables $\overline{\mathbf{g}}(s)\equiv(\overline{g_v},\overline{x_1},\overline{x_2},\overline{x_3})$ which are functions of the rescaled wave number $s=k/\Lambda$. The dependence on the scale of these effective variables is governed by the system of differential equations [11]

$$s\frac{d\overline{g_v}}{ds}=\beta_{g_v}(\overline{\mathbf{g}};\alpha_1,\alpha_2,d),$$

$$s\frac{d\overline{\chi_j}}{ds}=\beta_{\chi_j}(\overline{\mathbf{g}};\alpha_1,\alpha_2,d),\quad 0\leqslant s\leqslant 1 \tag{3.1}$$

with the initial conditions

$$\overline{\mathbf{g}}|_{(s=1)}=\mathbf{g}\equiv(g_v,\chi_1,\chi_2,\chi_3). \tag{3.2}$$

The large-scale limit of the statistical theory is described by the stable fixed points of the renormalization group determined by

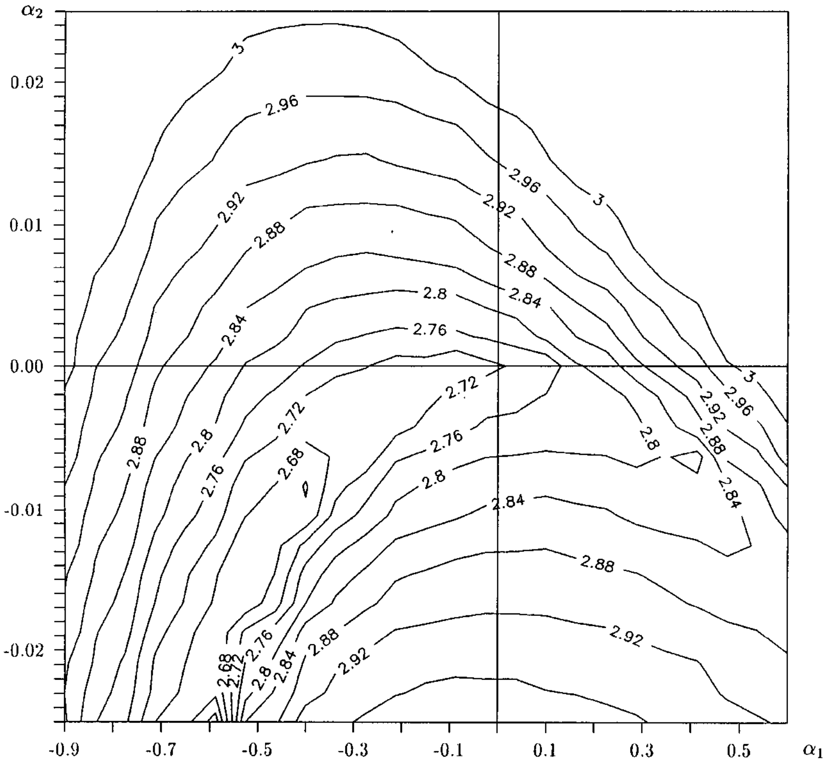

FIG. 1. Borderlines in the (α_1,α_2) plane of the region of stability of the Kolmogorov scaling regime are shown for several space dimensions $2<d\leqslant 3$. Despite the physical condition $\alpha_2>0$, to obtain a complete survey of the stability region we have included also the region of negative α_2.

$$\overline{\mathbf{g}}_{(s\to 0)}=\mathbf{g}^*,\quad \beta_{g_v}(\mathbf{g}^*;\alpha_1,\alpha_2,d)=0\,,$$

$$\beta_{\chi_j}(\mathbf{g}^*;\alpha_1,\alpha_2,d)=0\,. \tag{3.3}$$

At the one-loop order of the minimal subtraction scheme [9] the β functions on the right-hand side of Eqs. (3.1) may be expressed in terms of the renormalization constants as

$$\beta_{g_v}=2g_v\epsilon(1-3Z_1),$$

$$\beta_{\chi_j}=2\,\chi_j\epsilon(Z_{j+1}-Z_1),\quad j=1,2,3.$$

The explicit form of the complicated β_g and β_{χ_j} functions obtained from one-loop diagrams is presented in Appendix A.

The location and stability of the fixed points in the $\mathbf{g}$ space depend on the parameters d, α_1, and α_2. The expressions for the β functions contain integrals, which have to be calculated numerically. Therefore only a numerical analysis (see Appendix B) of the renormalization-group flows (3.1) may be used to clarify different aspects of the strong anisotropy problem. The set of flow patterns determined by the initial conditions (3.2) leading to a stable fixed point (3.3) constitute the universality class of this fixed point.

Numerical investigation illustrated in Figs. 1–9 confirms the existence of a universal kinetic scaling regime corresponding to a stable fixed point of the renormalization group for not too large values of the anisotropy parameters α_1 and α_2. The Kolmogorov spectral index of this fixed point remains fixed to a value independent of d (see Sec. IV). However, for large physically allowed values of the anisotropy parameters the Kolmogorov scaling regime becomes unstable, with a region of stability shrinking when the dimension of space is decreased.

Various aspects of axisymmetrically driven turbulence

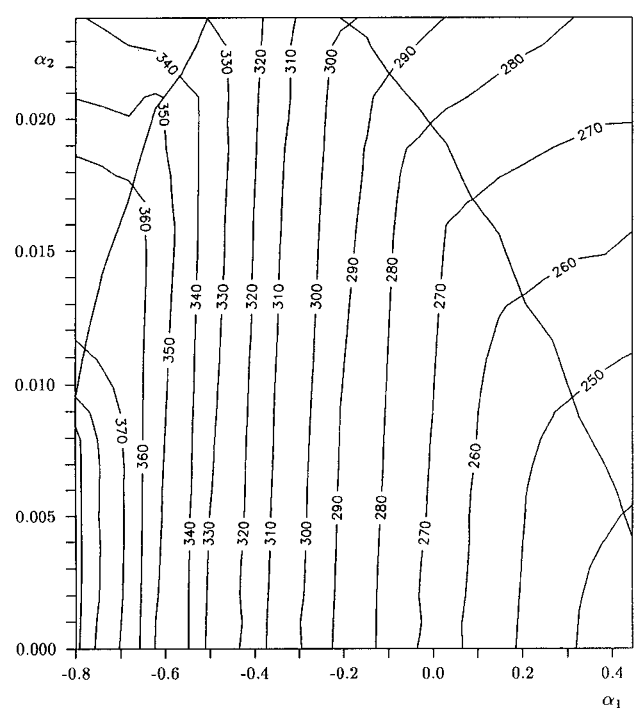

FIG. 2. Contour map of the fixed point values of the coupling constant $g_v^*(\alpha_1,\alpha_2)$ at $d=3$. The borderline of the stability region of the Kolmogorov scaling regime is also shown. Note the weak dependence of g_v^* on the parameter α_2 in the stability region.

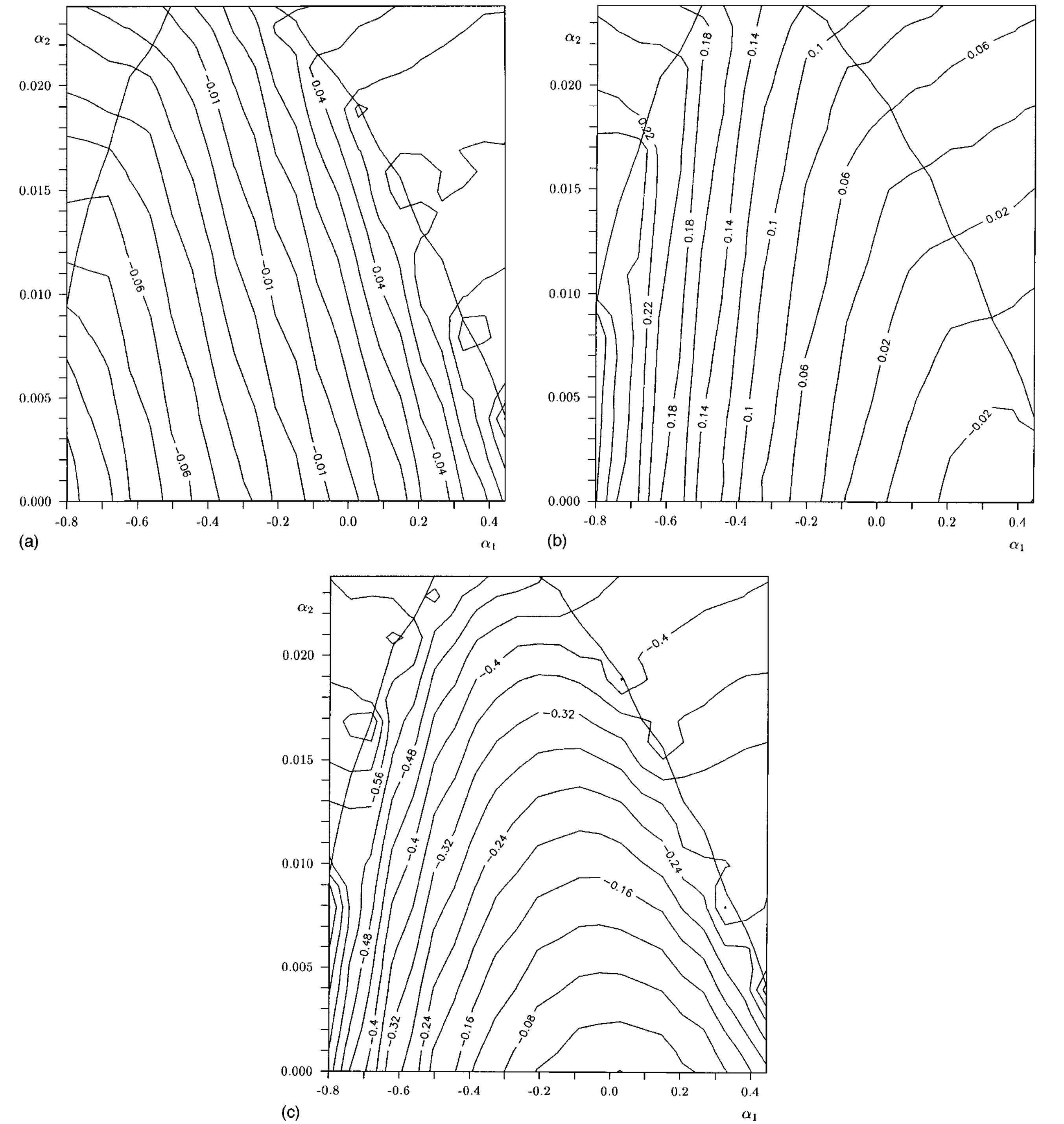

FIG. 3. Contour maps of the fixed point values of the parameters χ_1^*, χ_2^*, and χ_3^* of the effective viscosity tensor. (a) Contour map of the fixed point values of χ_1^*. Note the nearly linear behavior of the equal-value contours of χ_1^* in the stability region. (b) Contour map of the fixed point values of χ_2^*. (c) Contour map of the fixed point values of χ_3^*. Note that the strongly curved equal-value contours of χ_3^* in the stability region are qualitatively different from the straight-line contours of the parameters g_v^*, χ_1^*, and χ_2^*.

were discussed earlier in the weak-anisotropy approximation [3,4]. In the present work we analyze stability of the Kolmogorov scaling against strong anisotropy and calculate the dependence of the Kolmogorov constant on the anisotropy parameters. It is a key step of this formulation to assess the relevance of the term $\chi_3 \nu \vec{n}(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{v})$ in the effective anisotropic viscosity tensor, which is absent in the weak-anisotropy approximation of Rubinstein and Barton [3]. According to our investigation of Eq. (2.1) in the weak-anisotropy limit, the presence of nonzero χ_3 parameter is irrelevant for the stability of the theory at $d=3$, see Fig. 1, or eventually see the relations (4.15).

The turbulence is expected to be a universal phenomenon. However, the notion of universality must be clarified here, because all of the obtained fixed point values of the parameters $\mathbf{g}$ depend on the anisotropy parameters α_1 and α_2.

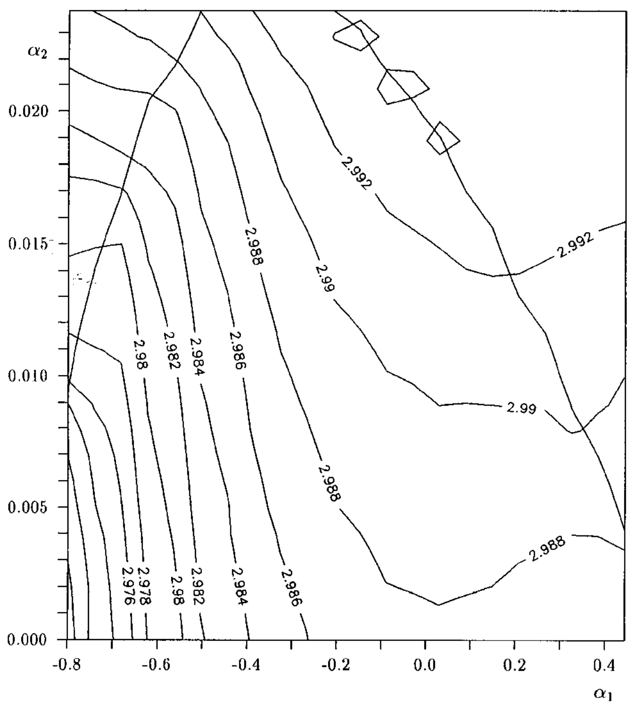

FIG. 4. Contour map of the Kolmogorov constant $C_k(\alpha_1,\alpha_2)$. Note that maximal changes of C_k caused by the anisotropy are of the order of $O(10^{-3})$.

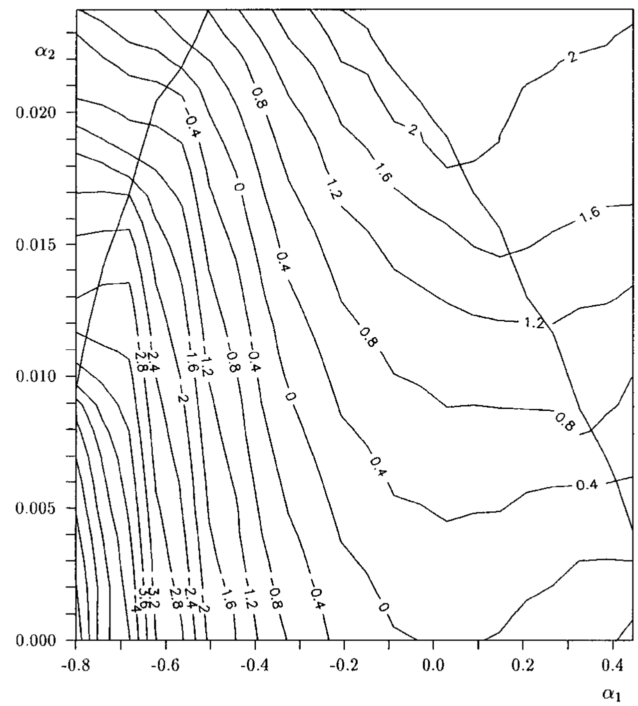

FIG. 6. Contour map of the function $10^3\times\sigma^{\mathrm{isotr}}(\alpha_1,\alpha_2)$, which characterizes relative difference of the effective Kolmogorov constant of the anisotropic model and the Kolmogorov constant of the isotropic model.

Therefore the plausible notion of universality can be fully related only to such characteristics, which remain unchanged, if the external axial symmetry lowers the symmetry of the turbulent system. In the case of the kinetic regime, an example of such a persistent variable is the spectral index $[\nu]=2\epsilon/3$, which determines the large-scale asymptotics of turbulent viscosity.

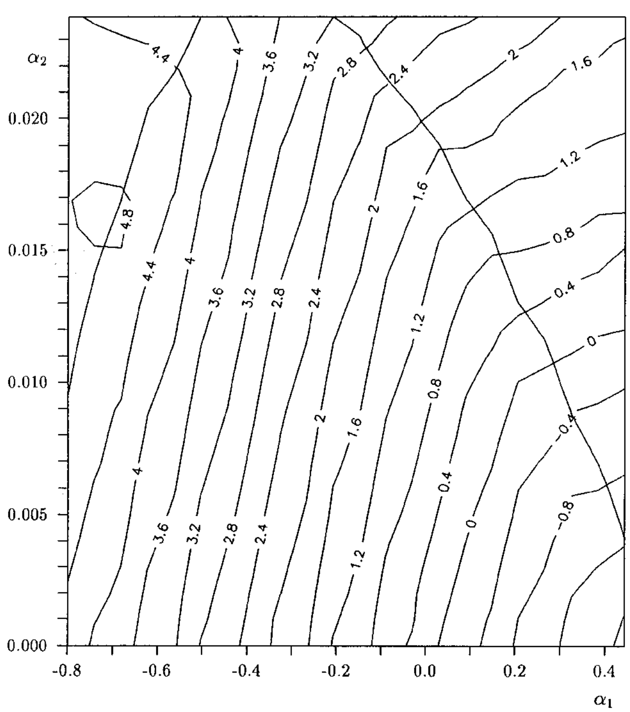

FIG. 5. Contour map of the function $10^2\times\sigma^{\parallel,\perp}(\alpha_1,\alpha_2)$, which characterizes relative difference of the longitudinal and transverse effective Kolmogorov constants.

Henceforth, we restrict ourselves to the presentation of the main results of the renormalization-group analysis, which are focused on the investigation of the Kolmogorov scaling regime with an emphasis on the d-dimensional approach $(2<d\leqslant 3)$.

Calculations on the strongly anisotropic system have shown that qualitative changes in anisotropically driven turbulence can take place at dimensions between two and three. For each fixed value of the dimension of space d there is a region in the (α_1,α_2) plane, in which the Kolmogorov scaling regime is stable against anisotropic forcing in the sense that there is an infrared-stable fixed point of the renormalization group with the Kolmogorov spectral index independent of d and the anisotropy parameters α_1 and α_2. Outside the stability region this fixed point is not infrared stable—numerical analysis indicates that the system approaches a strong-coupling regime there—and the Kolmogorov scaling regime does not exist. The borderline of the region of stability is shown in Fig. 1 for several values of the space dimensionality. It should be noted that at three dimensions anisotropic forcing destabilizes the Kolmogorov regime at rather small values of $\alpha_2>0.0235$. Figure 1 also illustrates the fact that the Kolmogorov regime becomes unstable against any anisotropy below a critical dimension [4]

$$d_c=\frac{3\sqrt{17}-7}{2}\simeq 2.6846, \qquad (3.4)$$

because the region of stability vanishes for physically al-

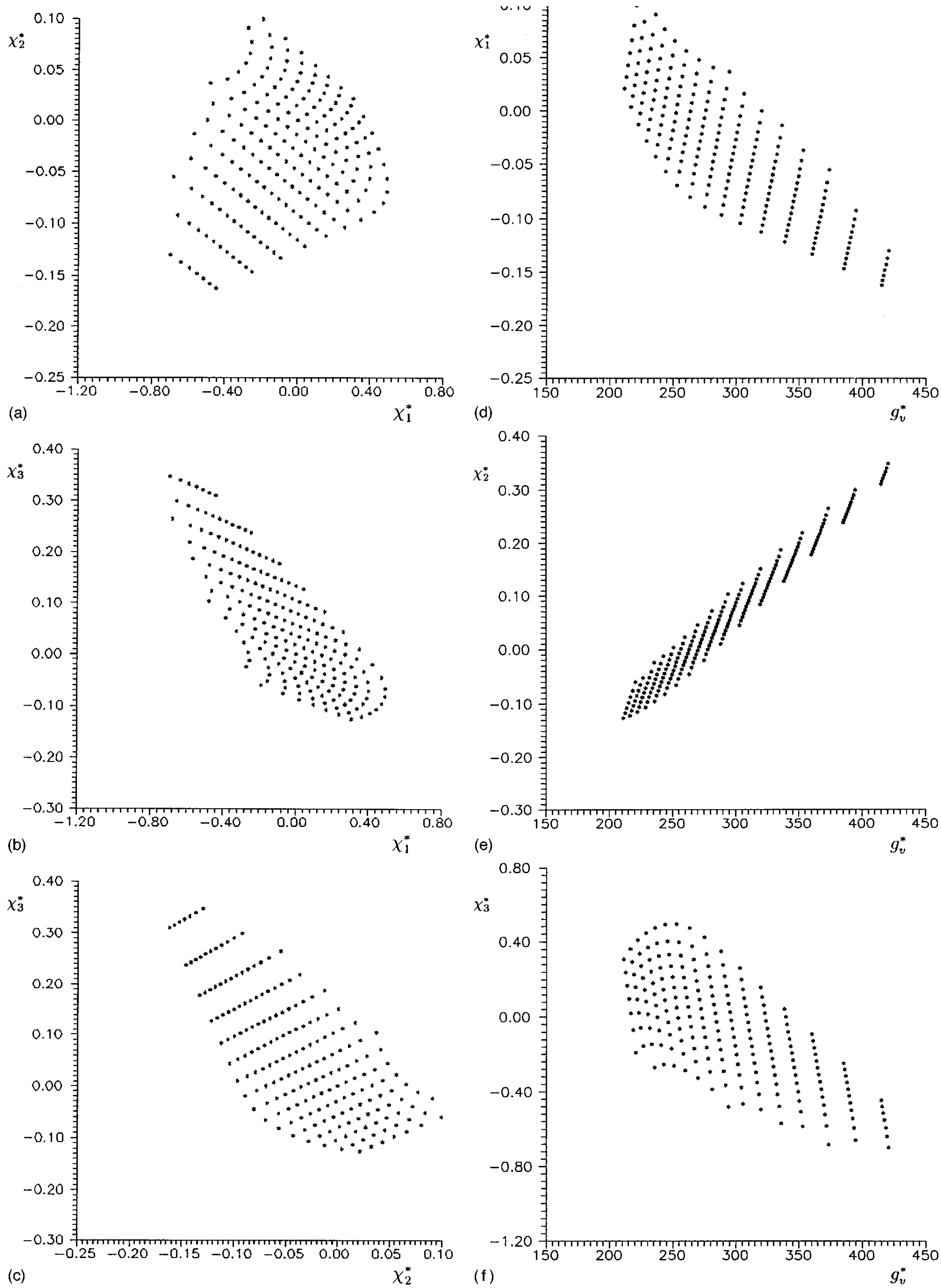

FIG. 7. Distribution of fixed point values of parameters calculated for a regular square mesh in the parametric space of α_1, α_2 . The calculation has been carried out only for stable fixed parameters. Data projections of obtained vectors $[\alpha_1, \alpha_2, g_v^*(\alpha_1,\alpha_2), \chi_1^*(\alpha_1,\alpha_2), \chi_2^*(\alpha_1,\alpha_2), \chi_3^*(\alpha_1,\alpha_2)]$ onto two-dimensional subspace of the parameters (χ_1^*, χ_2^*) are shown in (a), onto subspace of the parameters (χ_1^*, χ_3^*) in (b), onto subspace of the parameters (χ_2^*, χ_3^*) in (c), onto subspace of the parameters (g_v^*, χ_1^*) in (d), onto subspace of the parameters (g_v^*, χ_2^*) in (e), and onto subspace of the parameters (g_v^*, χ_3^*) in (f).

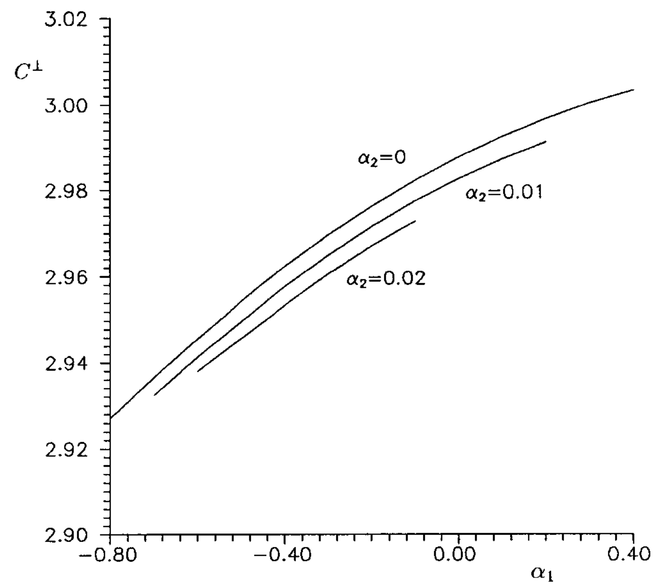

FIG. 8. Plot of the effective transverse Kolmogorov constant $C^\perp$ as a function of α_1 for different values of α_2.

lowed values of the anisotropy parameters $\alpha_1 \geqslant -1$, $\alpha_2 \geqslant 0$. Figure 1 also shows that the region of stability of the Kolmogorov scaling regime in the parametric space of α_1, α_2 decreases with the dimension of space.

IV. EFFECTIVE KOLMOGOROV CONSTANT

The basic parameter characterizing the inertial range energy cascade is the Kolmogorov constant C_k. Here we discuss the anisotropic theory, where the effective C_k is not constant, but depends on the anisotropy parameters α_1 and α_2 [or $\mathbf{g}^*$ through Eq. (3.1)]. The calculation of $C_k(\alpha_1,\alpha_2)$ is carried out following the main steps of the isotropic theory [12]. However, it is not possible to obtain analytic results here, and the dependence on α_1 and α_2 of the

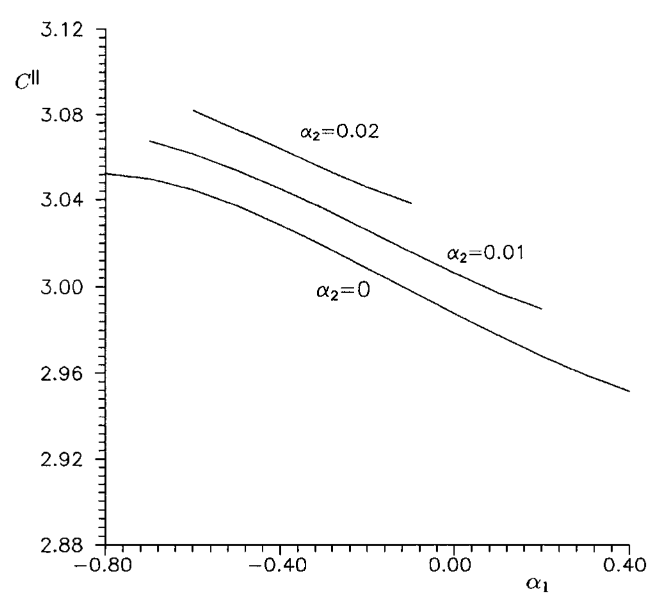

FIG. 9. Plot of the effective longitudinal Kolmogorov constant $C^{\parallel}$ as a function of α_1 for different values of α_2.

amplitudes of the longitudinal and transverse projection operators in the radial energy spectrum has been investigated numerically.

Taking into account the stationarity and translational invariance of the model, the correlation functions can be expressed in terms of relative coordinates. This implies that the equal-time velocity pair correlation function is given by

$$\langle v_j(\vec{x}_1,t)v_m(\vec{x}_2,t)\rangle = \int \frac{d^d\vec{k}}{(2\pi)^d} G^R_{jm}(\vec{k})\exp[i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)].$$

Here, the spatial Fourier transform $G^R_{jm}(\vec{k})$ as a function of renormalized variables

$$G^R_{jm}(\vec{k}) = \frac{1}{2}\nu^2\Lambda^{2\epsilon}k^{2-2\epsilon-d}U_{jm}(\overline{\mathbf{g}}(s))\Psi(s), \qquad (4.1)$$

with

$$\Psi(s) = s^{2\epsilon}\exp\left[-2\int_1^s \frac{d\tilde{s}}{\tilde{s}}\gamma_1(\overline{\mathbf{g}}(\tilde{s}))\right], \quad \gamma_1 = \frac{\partial \ln Z_1}{\partial \ln \Lambda} \qquad (4.2)$$

satisfies the renormalization-group equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta_{g_v}\frac{\partial}{\partial g_v} + \sum_{j=1}^{3}\beta_{\chi_j}\frac{\partial}{\partial\chi_j} - \gamma_1\nu\frac{\partial}{\partial\nu}\right]G^R_{jm}(\vec{k}) = 0\,.$$

The leading-order approximation of the function U_{jm} can be derived from the condition that at the lowest order of perturbative expansion the correlation function G^R must coincide with the propagator $\Delta_{jm}(\vec{k},\tau=0)$ of Eq. (2.9), therefore

$$\begin{aligned} U_{jm}(\overline{\mathbf{g}}(s)) &\approx \overline{g_v}(s)\Pi_{jm} \\ &\equiv \overline{g_v}(s)[P_{jm}(\vec{k})w_1(\xi_k,0) \\ &\quad + R_{jm}(\vec{k})w_2(\xi_k,0)]_{\chi_j\to\overline{\chi_j}(s)}\,. \end{aligned}$$

Using the differential equation (3.1) for the variable $\overline{g_v}(s)$ it can be shown that

$$\int_1^s \frac{d\tilde{s}}{\tilde{s}}\gamma_1(\overline{\mathbf{g}}(\tilde{s})) = \frac{1}{3}\ln\left(\frac{\overline{g_v}(s)}{g_v}\right) + \frac{2\epsilon}{3}\ln s. \qquad (4.3)$$

Thus from Eqs. (4.2) and (4.3) it follows that

$$\Psi(s) = s^{2\epsilon/3}\left(\frac{g_v}{\overline{g_v}(s)}\right)^{2/3}. \qquad (4.4)$$

Substitution of Eq. (4.4) into Eq. (4.1) yields the expression

$$G^R_{jm}(\vec{k}) = \frac{1}{2}\nu^2[g_v]^{2/3}[\overline{g_v}(s)]^{1/3}k^{2-d-4\epsilon/3}\Lambda^{4\epsilon/3}\Pi_{jm}(\overline{\mathbf{g}}(s)). \qquad (4.5)$$

The perturbatively exactly calculated universal large-scale asymptotics $k^{2-d-4\epsilon/3}$ in Eq. (4.5) can be considered the main result of the renormalization-group analysis.

Next we establish a relation between the mean dissipation rate of energy $\overline{\varepsilon}$ and the parameters g_v and ν. Comparing the Schwinger equation $\langle \delta S/\delta \tilde{v}_j \rangle = 0$ with Eq. (2.1) yields the relation

$$f_j(\vec{x},t) = \frac{1}{2}\int d^d\vec{x}_1 \int dt_1 \tilde{v}_l(\vec{x}_1,t_1)[D_{jl}(\vec{x}-\vec{x}_1,t-t_1) + D_{lj}(\vec{x}_1-\vec{x},t_1-t)], \quad (4.6)$$

which couples the auxiliary field $\tilde{\vec{v}}$ and the random force field $\vec{f}$ by means of the correlation matrix of the random force D_{jl}. Using Eqs. (2.2) and (4.6) and the symmetry property $D_{jl} = D_{lj}$, we obtain the correlation function

$$\langle \vec{f}(\vec{x},t)\cdot\vec{v}(\vec{x},t)\rangle = \int d^d\vec{x}_1 \int \frac{d^d\vec{k}}{(2\pi)^d}\tilde{\mathcal{D}}_{jl}(\vec{k})\cos[\vec{k}\cdot(\vec{x}-\vec{x}_1)] \times \langle v_j(\vec{x},t)\tilde{v}_l(\vec{x}_1,t)\rangle. \quad (4.7)$$

To derive an analytic formula for the correlation function $\langle \vec{f}\cdot\vec{v}\rangle$, we employ the connection (4.7) and the exact form of the equal-time response function $\langle v_j \tilde{v}_\alpha \rangle$. Each term of the perturbative expansion of the response function $\langle v_j(\vec{x},t)\tilde{v}_\alpha(\vec{x}_1,t)\rangle$ contains continuous chains of retarded $\tilde{\Delta}(k,t)$ propagators. In the equal-time response function the step functions $\theta(t)$ in $\tilde{\Delta}(k,t)$ form closed loops which produce vanishing time integrals, apart from the case of a single $\tilde{\Delta}(k,t)$ propagator [7]. The resulting expression

$$\langle v_j(\vec{x},t)\tilde{v}_\alpha(\vec{x}_1,t)\rangle = \int \frac{d^d\vec{k}}{(2\pi)^d}\tilde{\Delta}_{\alpha j}(\vec{k},0)\exp[i\vec{k}\cdot(\vec{x}-\vec{x}_1)]$$

contains only the remaining zeroth-order contribution. Using Eq. (2.9) we obtain

$$\langle v_j(\vec{x},t)\tilde{v}_\alpha(\vec{x}_1,t)\rangle = \frac{1}{2}\int \frac{d^d\vec{k}}{(2\pi)^d}P_{\alpha j}(\vec{k})\exp[i\vec{k}\cdot(\vec{x}-\vec{x}_1)], \quad (4.8)$$

where the convention $\theta(0) = 1/2$ has been used.

When Eq. (4.8) is combined with Eq. (4.7) the correlation function of random force and velocity field acquires the form

$$\langle \vec{f}(\vec{x},t)\cdot\vec{v}(\vec{x},t)\rangle = \frac{1}{2}\int_{k\leq\Lambda}\frac{d^d\vec{k}}{(2\pi)^d}\tilde{\mathcal{D}}_{jj}(\vec{k}), \quad (4.9)$$

where we have introduced the cutoff parameter Λ explicitly. Averaging the scalar product of Eq. (2.1) and $\vec{v}$, and taking into account that the terms $\partial_t\langle \vec{v}\cdot\vec{v}\rangle$, $\langle \vec{v}\cdot[(\vec{v}\cdot\vec{\nabla})\vec{v}]\rangle$ vanish due to stationarity and translational invariance we arrive at the energy balance condition

$$\overline{\varepsilon} \equiv \langle \vec{f}^A(\vec{x},t)\cdot\vec{v}(\vec{x},t)\rangle = \langle \vec{f}(\vec{x},t)\cdot\vec{v}(\vec{x},t)\rangle, \quad (4.10)$$

where $\overline{\varepsilon}$ is the mean rate of energy dissipation. From Eqs. (4.9) and (4.10) we have

$$\overline{\varepsilon} = \frac{1}{2}\int_{k\leq\Lambda}\frac{d^d\vec{k}}{(2\pi)^d}\tilde{\mathcal{D}}_{jj}(\vec{k}).$$

With the help of Eq. (2.3) and subsequent analytical integration the following explicit form of the mean rate of energy dissipation is derived:

$$\overline{\varepsilon} = g_v \nu^3 \frac{S_d(d-1)}{2(2\pi)^d}\frac{\Lambda^4}{4-2\epsilon}\left(1+\frac{\alpha_1+\alpha_2}{d}\right), \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad (4.11)$$

where $\Gamma(x)$ is the gamma function. Using Eq. (4.11) we can express ν^2 as a function of the product $\overline{\varepsilon}^{2/3} g_v^{-2/3}\Lambda^{-8/3}(4-2\epsilon)^{2/3}$ and substitute it in Eq. (4.5). In the large-scale limit we obtain

$$G_{jm}^{R(*)}(\vec{k}) = \overline{\varepsilon}^{2/3}\tilde{c}_a k^{2-d-4\epsilon/3}\Lambda^{(4/3)(\epsilon-2)}\left(1-\frac{\epsilon}{2}\right)^{2/3} \times (2g_v^*)^{1/3}\Pi_{jm}(\mathbf{g}^*), \quad (4.12)$$

with

$$\tilde{c}_a = \left(\frac{2(2\pi)^d d}{S_d(d-1)(d+\alpha_1+\alpha_2)}\right)^{2/3},$$

where the notation $G_{jm}^{R(*)}(\vec{k})$ means that G^R and the functions (2.10) are taken at the fixed point corresponding to the Kolmogorov scaling regime. In the framework of the ϵ-expansion method the coefficient of the powerlike wave-number dependence of the velocity correlation function is calculated up to the $O(\epsilon^{1/3})$ order, and the Kolmogorov values of the exponents $4(\epsilon-2)/3$ and $2-d-4\epsilon/3$ can be obtained by choosing for the expansion parameter the value $\epsilon = 2$. Thus we obtain from Eq. (4.12) the expression

$$G_{jm}^{R(*)}(\vec{k}) = {}^{2/3}c_a k^{-2/3-d}\Pi_{jm}(\mathbf{g}^*), \quad c_a = (2g_v^*)^{1/3}\tilde{c}_a .$$

In order to better understand the possible measurable consequences of the presence of anisotropy, we define the radial spectral tensor as

$$E_{jm}(k) = \frac{1}{2(2\pi)^d}k^{d-1}\int d\Omega_{\vec{k}} G_{jm}^{R(*)}(\vec{k}). \quad (4.13)$$

Here, $d\Omega_{\vec{k}}$ denotes the measure of integration over the spherical surface of radius $|\vec{k}|$ centered at $\vec{k} = \vec{0}$.

Instead of the Kolmogorov constant C_k of isotropic turbulence, we introduce two independent scalar amplitudes $C_\parallel$ and $C_\perp$, which decompose E_{jm} to longitudinal and transverse parts with respect to the preferred direction,

$$E_{jm}(k) = \frac{1}{d}[C_\perp P_{jm}(\vec{n}) + C_\parallel P_{jm}^\parallel(\vec{n})]\overline{\varepsilon}^{2/3}k^{-5/3}. \quad (4.14)$$

Using the expressions (4.13) and (4.14) with the subsequent extraction of the projection operators we obtain

$$C_\parallel = c_b\int_0^1 d\xi(1-\xi^2)^{(d-1)/2}[w_1^*(\xi,0) + (1-\xi^2)w_2^*(\xi,0)],$$

$$C_\perp = \frac{c_b}{d-1}\int_0^1 d\xi(1-\xi^2)^{(d-3)/2} \times [(d-2+\xi^2)w_1^*(\xi,0) + \xi^2(1-\xi^2)w_2^*(\xi,0)],$$

$$c_b = c_a S_{d-1} d/(2\pi)^d.$$

Substituting Eq. (2.10) in the preceding expressions we obtain

$$C_\parallel = c_b \int_0^1 d\xi (1-\xi^2)^{(d-1)/2} \mathcal{K}(\xi),$$

$$C_\perp = c_b \int_0^1 d\xi (1-\xi^2)^{(d-3)/2} \left((d-2)\frac{1+\alpha_1 \xi^2}{1+\chi_1^* \xi^2} + \xi^2 \mathcal{K}(\xi) \right),$$

with the function

$$\mathcal{K}(\xi) = \frac{1+\alpha_2 + (\alpha_1 - \alpha_2)\xi^2}{1+\chi_2^* + (\chi_1^* - \chi_2^* + \chi_3^*)\xi^2 - \chi_3^* \xi^4}.$$

Comparison of the trace of the matrix (4.14) with the definition of the isotropic Kolmogorov constant

$$E_{jj}(k) = C_k \overline{\varepsilon^{2/3}} k^{-5/3}$$

yields

$$C_k = \frac{(d-1)C_\perp + C_\parallel}{d}.$$

The Kolmogorov constant C_k and the parameters $C_\perp$ and $C_\parallel$ may be determined using the fixed point parameters of the renormalization group calculated numerically.

We define the relative measures $\sigma^{\parallel,\perp}$ and σ^{isotr} characterizing deviation of the system from the isotropic state as

$$\sigma^{\parallel,\perp} = \frac{C^\parallel - C^\perp}{C_k}, \quad \sigma^{\text{isotr}} = \frac{C_k - C_k^{\text{isotr}}}{C_k^{\text{isotr}}},$$

where C_k^{isotr} is the Kolmogorov constant in the isotropic stochastic model [12].

Let us consider the most important case $d=3$, which is well illustrated in Figs. 2–9. For $d=3$ the Kolmogorov constant of isotropic system $C_k^{\text{isotr}} = (80/3)^{1/3} \simeq 2.9876$ [12]. Analyzing available data we have found that the values of the parameters α_1, α_2 fall into the region of stability of Kolmogorov scaling if they satisfy the inequalities

$$0 < \alpha_2 < 0.017\,68 - 0.029\,12\alpha_1 - 0.023\,85\alpha_1^2 + 0.032\,84\alpha_1^3,$$
$$-0.906 < \alpha_1 < 0.546. \tag{4.15}$$

We have also found that in the stability region the anisotropy measures $\sigma^{\parallel,\perp}$, σ^{isotr} are subject to the following limitations (Figs. 5–6):

$$-1.7\times 10^{-2} < \sigma^{\parallel,\perp} < 5.04\times 10^{-2},$$

$$-6.2\times 10^{-3} < \sigma^{\text{isotr}} < 2.4\times 10^{-3}.$$

For the numerical values of C_k and related parameters we find (Figs. 4–6, 8, and 9) that

$$2.97 < C_k < 2.995, \quad 2.951 < C^\parallel < 3.082, \quad 2.925 < C^\perp < 3.003.$$

It turns out that the effect of induced anisotropy is hardly visible in the calculated value of the parameter C_k. However, our predictions of the variations of $\sigma^{\parallel,\perp}$ and spectral ''splitting'' shown in Figs. 8 and 9, which are of the order of about a few percent allow us to speculate about experimental verification of these results. The analysis has revealed that the radial energy spectrum including the information about the statistics of velocity modes is not sensitive to the presence of anisotropy. It can be seen from Figs. 7(a)–7(f) that the effect of anisotropy is more pronounced in the variations of the fixed point values of the parameters

$$-0.11 < \chi_1^* < 0.1, \quad -0.04 < \chi_2^* < 0.31, \quad -0.72 < \chi_3^* < 0,$$

which describe the effective anisotropic viscosity.

V. ANISOTROPIC RESISTIVE AND LORENTZIAN FORCES IN STOCHASTIC MAGNETOHYDRODYNAMICS

There are several mechanisms through which the MHD turbulent media become anisotropic. The anisotropy can arise in the presence of a uniform background magnetic field [13], macroscopic polarization of the turbulent media, or anisotropy induced by specific random forcing [5]. The last example deals with the problem of how the anisotropic forcing determines the inertial properties of the MHD.

The initial object of our treatment—stochastic anisotropic hydrodynamics—is used to construct a renormalizable theory of randomly driven anisotropic magnetohydrodynamics. The equations of the anisotropically driven MHD fluid [5] (under the conditions of very high Reynolds numbers and magnetic Reynolds numbers) may be written as

$$\partial_t \vec{v} + \hat{P}[(\vec{v}\cdot\vec{\nabla})\vec{v} + (\vec{b}\cdot\vec{\nabla})\vec{b}] - \nu \vec{\nabla}^2 \vec{v} - \vec{f}^A + \vec{f}^L = \vec{f},$$

$$\partial_t \vec{b} + (\vec{v}\cdot\vec{\nabla})\vec{b} - (\vec{b}\cdot\vec{\nabla})\vec{v} - u\nu \vec{\nabla}^2 \vec{b} - \vec{f}^B = \vec{f}^b,$$

$$\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot\vec{b} = \vec{\nabla}\cdot\vec{f} = \vec{\nabla}\cdot\ \vec{f}^b = 0,$$

where u is the dimensionless inverse magnetic Prandtl number. The correlations of the magnetic forcing satisfy the usual assumption of uncorrelated $\vec{f}^b$ and $\vec{f}$ forces. The statistics of the magnetic random force is assumed to be Gaussian with the correlation functions [5]

$$\langle f_j^b(\vec{x}_1,t_1) f_s(\vec{x}_2,t_2)\rangle = 0,$$

$$\langle f_j^b(\vec{x}_1,t_1) f_s^b(\vec{x}_2,t_2)\rangle = D_{js}^b(\vec{x}_1 - \vec{x}_2, t_1 - t_2).$$

The correlation function D_{js}^b is defined by the relations (2.2) and (2.3) and the following reparametrization $g_v \to g_b$, $\alpha_1 \to \alpha_3$, $\alpha_2 \to \alpha_4$, $\epsilon \to \epsilon'$. Here, α_3, α_4, and the exponent ϵ' are additional free parameters and g_b is a new coupling parameter. The anisotropy of the magnetic forcing requires appropriate additional terms with lower symmetry to be added to the model, which are absent in the standard MHD equations. The anisotropic magnetodissipative term analogous to (2.7) is

$$\vec{f}^B = u\nu[\chi_4(\vec{n}\cdot\vec{\nabla})^2\vec{b} + \chi_5 \hat{P}\vec{n}\vec{\nabla}^2(\vec{n}\cdot\vec{b}) + \chi_6 \hat{P}\vec{n}(\vec{n}\cdot\vec{\nabla})^2(\vec{n}\cdot\vec{b})],$$

where $\chi_j, j=4,5,6$ are new dimensionless parameters. All the terms, which are generated in the process of renormalization, must be included in the definition of the modified Lorentzian force in order to obtain a multiplicatively renormalizable model. The additional terms can be written in the form

$$\vec{f}^L=\hat{P}[\lambda_1\vec{b}(\vec{n}\cdot\vec{\nabla})(\vec{n}\cdot\vec{b})+\lambda_2\vec{n}(\vec{n}\cdot\vec{\nabla})b^2+\lambda_3\vec{n}(\vec{b}\cdot\vec{\nabla})(\vec{n}\cdot\vec{b})$$
$$+\lambda_4\vec{n}(\vec{n}\cdot\vec{b})(\vec{n}\cdot\vec{\nabla})(\vec{n}\cdot\vec{b})]$$

regarded as a modification of the isotropic Lorentzian term $[\vec{b}\cdot\vec{\nabla}]\vec{b}$.

Recently, a renormalizable variant of the MHD theory with simpler terms $\vec{f}^L_{\lambda_4\to 0}$, $\vec{f}^A|_{\chi_3\to 0}$, $\vec{f}^B|_{\chi_6\to 0}$ has been studied in detail [5]. This treatment was limited by the assumption that the anisotropy parameters $\alpha_l, l=1,2,3,4$ and consequently the fixed point values $\chi_j^*, j=1,2,4,5; \lambda_i^*, i=1,2,3$ were small. It was shown that the investigation of MHD in the weak anisotropy limit leads to Kolmogorov's spectral prediction (for the kinetic and the magnetic energy spectra) if the forcing exponents satisfy the physically unacceptable inequality $\epsilon'<0.65\epsilon$. The results obtained in the hydrodynamic theory with strong anisotropy allow us to speculate about the consequences for the stability of the MHD.

We believe that the presence of the χ_6 magnetodissipative term, analogous to the previously discussed viscous χ_3 term, will have immediate implications for the large-scale effect of the anisotropic $\vec{f}^L$ term and therefore will play an important role in the stabilization (eventually destabilization) of the critical regimes in the MHD.

VI. CONCLUSION

In this paper a renormalization-group analysis of anisotropically driven hydrodynamic turbulence has been carried out at the leading nontrivial one-loop order of the perturbation theory. The problem of the existence and stability of the Kolmogorov scaling regime has been investigated through numerical solution of the equations for the fixed point of the renormalization group. The principal conclusion from the results is that the Kolmogorov scaling regime can become unstable due to relatively small anisotropy variations; the less the space dimensionality, the more pronounced this property becomes. In particular, it has been confirmed that there is a critical dimension $d_c=(3\sqrt{17}-7)/2\simeq 2.6846$, below which the Kolmogorov scaling regime does not exist in the presence of anisotropic forcing. For the parameters describing the anisotropy limits have been established in which they give rise to the Kolmogorov scaling regime, for spatial dimensions $d_c\leqslant d\leqslant 3$. Relevance of a previously neglected term of the effective anisotropic viscous force has been discussed. An anisotropically driven MHD theory with similar strong-anisotropy terms has been put forward.

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APPENDIX A: CALCULATION OF β FUNCTIONS

In order to reduce as much as possible the number of terms arising from the integral (2.8) in the various computational stages, a few auxiliary identities and relations have been established. It is necessary to specify the form in which the functions $w_3(\xi_{k-p},\tau_{k-p})$, $w_4(\xi_{k-p},\tau_{k-p})$ and also the operators $P(\vec{k}-\vec{p})$ and $R(\vec{k}-\vec{p})$ will be expanded in the Taylor series in powers of $\vec{k}$ [external wave vector of the integral (2.8)]. The proportionality $V_{a\alpha\gamma}(\vec{k})\sim k$ is the reason, why in the final extraction of quadratic terms from (2.8) the Taylor expansion of the product $\tilde{\Delta}_{\gamma\delta}(\vec{k}-\vec{p},t)V_{\delta b\beta}(\vec{k}-\vec{p})$ is needed up to the first-order in $\vec{k}$ only. Let us define the coefficients $W_{(j,i)}$, $i=0,1,2,3$ (they depend on the scalar parameter ξ_p) as the first-order derivatives at vanishing scalar products $\vec{k}\cdot\vec{n}$ and $\vec{k}\cdot\vec{p}$,

$$W_{(j,0)}=w_j|_{(\vec{k}\cdot\vec{n};\vec{k}\cdot\vec{p}\to 0)},$$

$$W_{(j,1)}=p\frac{\partial w_j}{\partial(\vec{k}\cdot\vec{n})}|_{(\vec{k}\cdot\vec{n};\vec{k}\cdot\vec{p}\to 0)},$$

$$W_{(j,2)}=p^2\frac{\partial w_j}{\partial(\vec{k}\cdot\vec{p})}|_{(\vec{k}\cdot\vec{n};\vec{k}\cdot\vec{p}\to 0)}.$$

Then the expansion of w_3, w_4 is

$$w_j=W_{(j,0)}+W_{(j,1)}\left(\frac{\vec{k}\cdot\vec{n}}{p}\right)+W_{(j,2)}\left(\frac{\vec{k}\cdot\vec{p}}{p^2}\right) \qquad \text{(A1)}$$

for $j=3, 4$. Using the relations (2.9) we obtain the coefficients

$$W_{(3,1)}=2W_{(3,0)}\tau\xi\chi_1, \quad W_{(3,2)}=2\tau(A-\xi^2\chi_1)e^{-A\tau},$$

$$W_{(4,1)}=\frac{2e^{-A\tau-B\tau}}{(A-B)^2}(-B'e^{A\tau}M\xi+B'e^{B\tau}M\xi+AB'e^{A\tau}M\tau\xi-BB'e^{A\tau}M\tau\xi+e^{A\tau}M\xi\chi_1-e^{B\tau}M\xi\chi_1$$
$$-Ae^{B\tau}M\tau\xi\chi_1+Be^{B\tau}M\tau\xi\chi_1-Ae^{A\tau}\xi\chi_3+Be^{A\tau}\xi\chi_3+Ae^{B\tau}\xi\chi_3-Be^{B\tau}\xi\chi_3),$$

$$W_{(4,2)}=\frac{2e^{-A\tau-B\tau}}{(A-B)^2}(ABe^{A\tau}M\tau-B^2e^{A\tau}M\tau-A^2e^{B\tau}M\tau+ABe^{B\tau}M\tau+B'e^{A\tau}M\xi^2-B'e^{B\tau}M\xi^2-AB'e^{A\tau}M\tau\xi^2$$

$$+BB'e^{A\tau}M\tau\xi^2-e^{A\tau}M\xi^2\chi_1+e^{B\tau}M\xi^2\chi_1+Ae^{B\tau}M\tau\xi^2\chi_1-Be^{B\tau}M\tau\xi^2\chi_1+Ae^{A\tau}\xi^2\chi_3-Be^{A\tau}\xi^2\chi_3-Ae^{B\tau}\xi^2\chi_3$$
$$+Be^{B\tau}\xi^2\chi_3),$$

where

$$B'=\frac{\partial B}{\partial \xi^2}=\chi_1-\chi_2+\chi_3(1-\xi^2).$$

For the operators P and R as functions of the wave vector $\vec{k}-\vec{p}$ we find the expansion rules

$$P_{js}(\vec{k}-\vec{p})=P_{js}(\vec{p})-2p^{-4}(\vec{k}\cdot\vec{p})p_jp_s+p^{-2}(k_sp_j+k_jp_s),$$

$$R_{js}(\vec{k}-\vec{p})=R_{js}(\vec{p})+(n_j-p^{-1}\xi_pp_j)[p^{-1}\xi_pk_s+p^{-2}k\xi_kp_s-2p^{-3}\xi_p(\vec{k}\cdot\vec{p})p_s]$$
$$+(n_s-p^{-1}\xi_pp_s)[p^{-1}\xi_pk_j+p^{-2}k\xi_kp_j-2p^{-3}\xi_p(\vec{k}\cdot\vec{p})p_j].$$

In the following we present a procedure which considerably reduces the computational effort and replaces the integration with the measure $d^d\vec{p}$ by single integral over the variable ξ. The final step of the procedure—the one-dimensional integration over ξ—must be realized numerically. Let us denote by $F(\,)$ an arbitrary function of the argument $\vec{p}\cdot\vec{n}/p$. The following rules for the extraction of the $1/\epsilon$ poles in the minimal subtraction procedure [9]:

$$\int\frac{d^d\vec{p}}{(2\pi)^d}F(\xi_p)\frac{p_ip_jp_sp_m}{p^{4+d+2\epsilon}}=\frac{1}{2\epsilon}[I_{(4,1)}\{F\}n_in_jn_sn_m+I_{(4,2)}\{F\}(n_in_j\delta_{sm}+n_in_s\delta_{jm}+n_in_m\delta_{js}+n_jn_s\delta_{im}+n_jn_m\delta_{is}+n_sn_m\delta_{ij})$$
$$+I_{(4,3)}\{F\}(\delta_{ij}\delta_{sm}+\delta_{is}\delta_{jm}+\delta_{im}\delta_{sj})],$$

$$\int\frac{d^d\vec{p}}{(2\pi)^d}F(\xi_p)\frac{p_ip_jp_s}{p^{3+d+2\epsilon}}=\frac{1}{2\epsilon}[I_{(3,1)}\{F\}n_in_jn_s+I_{(3,2)}\{F\}(n_i\delta_{js}+n_j\delta_{is}+n_s\delta_{ij})],$$

$$\int\frac{d^d\vec{p}}{(2\pi)^d}F(\xi_p)\frac{p_ip_j}{p^{2+d+2\epsilon}}=\frac{1}{2\epsilon}[I_{(2,1)}\{F\}n_in_j+I_{(2,2)}\{F\}\delta_{ij}],$$

$$\int\frac{d^d\vec{p}}{(2\pi)^d}F(\xi_p)\frac{p_i}{p^{1+d+2\epsilon}}=\frac{1}{2\epsilon}n_iI_{(1,1)}\{F\},\quad \int\frac{d^d\vec{p}}{(2\pi)^d}F(\xi_p)\frac{1}{p^{d+2\epsilon}}=\frac{1}{2\epsilon}I_0\{F\}$$

are appropriate for the separation of the divergent part from the one-loop Feynman integral (2.8). Here, $I_{(X,Y)}\{F\}$ are linear in F functionals connected with the basic functional

$$I_0\{F\}\equiv I_{(0,0)}\{F\}=\frac{S_{d-1}}{(2\pi)^d}\int_{-1}^{1}d\xi(1-\xi^2)^{(d-3)/2}F(\xi) \tag{A2}$$

by means of the relations

$$I_{(4,1)}\{F\}=(d_4d_2I_0\{F\xi^4\}-6d_2I_0\{F\xi^2\}+3I_0\{F\})(d^2-1)^{-1},\quad I_{(4,2)}\{F\}=(-d_2I_0\{F\xi^4\}+d_3I_0\{F\xi^2\}-I_0\{F\})(d^2-1)^{-1},$$

$$I_{(4,3)}\{F\}=(I_0\{F\xi^4\}-2I_0\{F\xi^2\}+I_0\{F\xi^2\})(d^2-1)^{-1},\quad I_{(3,1)}\{F\}=(I_0\{F\xi^3\}d_2-3I_0\{F\xi\})(d-1)^{-1},$$

$$I_{(3,2)}\{F\}=(-I_0\{F\xi^3\}+I_0\{F\xi\})(d-1)^{-1},\quad I_{(2,1)}\{F\}=(I_0\{F\xi^2\}d-I_0\{F\})(d-1)^{-1},$$

$$I_{(2,2)}\{F\}=(I_0\{F\}-I_0\{F\xi^2\})(d-1)^{-1},\quad I_{(1,1)}\{F\}=I_0\{F\xi\},$$

where we have used the notation

$$d_j=d+j,\quad j=2,3,4.$$

In order to keep the number of terms in the renormalization constants Z_j of the renormalized action (2.10) as small as possible, it is useful to define the following $J_{a_1a_2b_1b_2c_1c_2}$ tensor structure:

$$J_{a_1a_2b_1b_2c_1c_2}=\frac{g_v}{4\epsilon}I_{(a_1,a_2)}\left\{\xi^{b_1}\int_{-\infty}^{\infty}d\tau w_{b_2}(\xi,\tau)W_{(c_1,c_2)}(\xi,\tau)\right\}. \tag{A3}$$

The expressions (A1) and (A3) may be used to bring the coefficients $Z_1,\ldots,Z_4$ into the compact form

$$\begin{aligned}
Z_1-1=&-J_{222140}-J_{222240}+J_{224240}-2J_{430130}-J_{430132}+4J_{432140}+J_{432142}+2J_{432230}+J_{432232}+2J_{432240}+J_{432242}\\
&-4J_{434240}-J_{434242},
\end{aligned}$$

$$\begin{aligned}
Z_2-1=&-J_{221141}-J_{221231}-J_{221241}-2J_{222240}+J_{223241}-J_{320131}-3J_{321140}-J_{321142}-2J_{321230}-J_{321232}-J_{321240}\\
&-J_{321242}+J_{322141}+J_{322231}+J_{322241}+5J_{323240}+J_{323242}-J_{324241}-2J_{420130}-J_{420132}+4J_{422140}+J_{422142}\\
&+2J_{422230}+J_{422232}+2J_{422240}+J_{422242}-4J_{424240}-J_{424242},
\end{aligned}$$

$$\begin{aligned}
Z_3-1=&J_{112140}+J_{112240}-J_{114240}-J_{212140}-J_{212240}+J_{214240}-J_{222240}-3J_{321140}-J_{321142}-2J_{321230}-J_{321232}\\
&-2J_{321240}-J_{321242}+5J_{323240}+J_{323242}-2J_{420130}-J_{420132}+4J_{422140}+J_{422142}+2J_{422230}+J_{422232}+2J_{422240}\\
&+J_{422242}-4J_{424240}-J_{424242},
\end{aligned}$$

$$\begin{aligned}
Z_4-1=&-J_{000240}+J_{111141}+J_{111231}+J_{111241}+5J_{112240}-J_{113241}+2J_{210140}+J_{210142}+2J_{210230}+J_{210232}+J_{210240}\\
&+J_{210242}-2J_{211141}-2J_{211231}-2J_{211241}-10J_{212240}-J_{212242}+2J_{213241}-J_{310131}-6J_{311140}-2J_{311142}\\
&-4J_{311230}-2J_{311232}-3J_{311240}-2J_{311242}+J_{312141}+J_{312231}+J_{312241}+10J_{313240}+2J_{313242}-J_{314241}-2J_{410130}\\
&-J_{410132}+4J_{412140}+J_{412142}+2J_{412230}+J_{412232}+2J_{412240}+J_{412242}-4J_{414240}-J_{414242}.
\end{aligned}$$

APPENDIX B: NUMERICAL METHODS USED IN THE SOLUTION OF DIFFERENTIAL EQUATIONS OF THE RENORMALIZATION GROUP

The system of differential equations (3.1) was solved numerically, using the fourth-order Runge-Kutta method with the adaptive choice of the integration step. For the variable $z=-\ln s$ the first step value $\Delta z=0.001$ was chosen. The step Δz was considered satisfactory, if the relative error of two consequent approximations did not exceed 10^{-5}. Our primary goal was to test the stability of the Kolmogorov scaling regime against anisotropic perturbations, therefore we used the fixed point of the three-dimensional isotropic model as the initial value $\mathbf{g}|_{z=0}=(g^*_{3D},0,0,0)$ for the solution. Physically, the initial values $\chi_i=0$ correspond to the assumed absence of anisotropy at small spatial scales. The independence of the fixed point $\mathbf{g}^*$ of the choice of the initial value $\mathbf{g}|_{z=0}$ was tested for selected points taken in the vicinity of the border of the region of stability, where the calculation of the fixed point parameters was repeated for randomly chosen initial value.

We compared two methods to deal with the singularities at $\xi=\pm 1$ for $d<3$ in the integrals (A2): the substitution $\xi=\sin\theta$ with the subsequent use of Simpson's rule, and the Chebyshev quadrature formula for the evaluation of integrals with the structure $a(\xi)/\sqrt{1-\xi^2}$, where $a(\xi)$ is a regular function. According to our experience the application of the Chebyshev method was more effective for the calculation of integrals (A2). In the numerical integration the division to 128 subintervals was used.

It must be emphasized that during the numerical calculation of the integrals, repeatedly applied on each step of the Runge-Kutta method, it was important to test simultaneously the conditions

$$\overline{\chi_1}>-1\,,\quad \overline{\chi_2}>-1\,,\quad \overline{\chi_3}>-(\sqrt{1+{}_1}+\sqrt{1+{}_2})^2. \tag{B1}$$

These were analytically derived from the requirement of convergence of the integrals (A2). We found out that a wide variety of unstable renormalization-group trajectories tend to violate the conditions (B1). At the beginning the evolution of trajectory starts as a long-lasting movement in the vicinity of the surface $\overline{\chi_3}=-(\sqrt{1+\overline{\chi_1}}+\sqrt{1+\overline{\chi_2}})^2$. After this period a quite rapid final expansion of the trajectories towards $||\mathbf{g}(s)||\to\infty$ follows. Therefore the numerical test based on the criteria (B1) represents an important test of the stability of the fixed point.

All calculations were performed on a 15×15 mesh in the space of parameters α_1 and α_2. To determine the boundary value of the space dimension d, the bisection method was used with the accuracy of 0.005 in the evaluation of d.

[1] L.W. Browne, R.A. Antonia, and D.A. Shah, J. Fluid Mech. **179**, 307 (1987).

[2] J.R. Herring, Phys. Fluids **17**, 859 (1974).

[3] R. Rubinstein and J.M. Barton, Phys. Fluids **30**, 2987 (1987).

[4] T.L. Kim and A.V. Serdyukov, Teor. Mat. Fiz. **105**, 412 (1995).

[5] L.Ts. Adzhemyan, M. Hnatich, D. Horvath, and M. Stehlik, Int. J. Mod. Phys. B **9**, 3401 (1995).

[6] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1989).

[7] L.Ts. Adzhemyan, A.N. Vasil'ev, and Yu.M. Pismak, Teor. Mat. Fiz. **57**, 268 (1983).

[8] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).

[9] G.'t Hooft, Nucl. Phys. B **61**, 455 (1973).
[10] J. Honkonen and M. Yu. Nalimov, Z. Phys. B **99**, 297 (1996).
[11] N. N. Bogoljubov and D.V. Shirkov, Doklady Akad. Nauk SSSR **103**, 203 (1955); Nuovo Cimento **3**, 845 (1956).
[12] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasil'ev, Zh. Eksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].
[13] M. Garnier, A. Alemany, P.L. Sulem, and A. Pouquet, J. Mec. **20**, 233 (1981).